Tutorial 1

Matrices

A rectangular array of numbers consisting of m rows and n columns is called a matrix

→ The numbers in the matrix are called entries

→ The order of a matrix with m rows and n columns is $m \times n$.

Columns

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$$

Rows

The order of A is 2×3.

In a matrix, an entry is often denoted by a_{ij} (or any letter with the subscript ij) where i and j are numbers, $1 \leq i \leq m$, $1 \leq j \leq n$.

So, in the above matrix A,

$$a_{11} = 1, \quad a_{12} = 4, \quad a_{13} = 7$$
$$a_{21} = 2, \quad a_{22} = 5, \quad a_{23} = 8$$

Types of matrices

1) Row matrix: Matrix with only one row ($m=1$)

eg: $\begin{bmatrix} 2 & 4 & 8 \end{bmatrix}$ order: 1×3

2) Column matrix: Matrix with only one column ($n=1$)

eg: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ order: 2×1

3) Rectangular matrix : Matrix with m columns and n rows, $m \neq n$.

eg: $\begin{bmatrix} 1 & 4 & 3 & 7 \\ 2 & 5 & 8 & 6 \end{bmatrix}$ Order : 2×4

4) Square Matrix : Rectangular matrix, but with $m = n$.

eg: $\begin{bmatrix} 1 & 9 & 2 \\ 2 & 7 & 1 \\ 3 & 8 & 5 \end{bmatrix}$ Order : 3×3

5) Diagonal matrix : Only the main diagonal can have non zero elements. Mathematically, $a_{ij} = 0$ if $i \neq j$.

eg: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ → only diagonal elements can be nonzero.

Note : $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is also diagonal

Order : 3×3

6) Upper triangular matrix : Only the main diagonal and every element above the diagonal can be nonzero. That is, $a_{ij} = 0$ for $i > j$.

eg: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & 10 \\ 0 & 0 & 7 \end{bmatrix}$

7) Lower Triangular matrix: The opposite of upper triangular. Only the main diagonal and every element below the diagonal can be non zero.
That is, $a_{ij} = 0$ for $i < j$.

eg: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 0 & 10 & 7 \end{bmatrix}$

8) Zero matrix: All entries of the matrix are zero.

eg: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

9) Identity matrix: Diagonal matrix, but with all the diagonal entries $= 1$.
That is

$$a_{ij} = 1 \quad \text{if } i = j$$
$$a_{ij} = 0 \quad \text{if } j \neq i$$

eg: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

We often denote them as I_n where n is the number of columns (which is equal to number of rows).

Trace: Denoted by $tr(A)$ or $sp(A)$, it is the sum of the entries on the main diagonal

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad tr(A) = 1+5+9 = 15$$

Only defined for sq. matrices.

Matrix Operations

1. Transposition

Inter changing rows and columns.

Simply stated, $a_{ij} \rightarrow a_{ji}$.

$$A \begin{bmatrix} 2 & 4 & 8 \\ 6 & 12 & 18 \end{bmatrix}, \quad \text{then } A^T = \begin{bmatrix} 2 & 6 \\ 4 & 12 \\ 8 & 18 \end{bmatrix}$$

If $A^T = A$, A is called symmetric.

eg: Any diagonal square matrix,

$$B = \begin{bmatrix} 10 & 80 \\ 80 & 10 \end{bmatrix}, \quad B^T = \begin{bmatrix} 10 & 80 \\ 80 & 10 \end{bmatrix}$$

Properties

1. $(A^T)^T = A$
2. $(A \pm B)^T = A^T \pm B^T$
3. $(KA)^T = KA^T$, K is scalar constant
4. $(A \cdot B)^T = B^T A^T$

2. Addition

If A, B have same order, then,

$C = A \pm B \Rightarrow c_{ij} = a_{ij} \pm b_{ij}$

Eg: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$A+B = \begin{bmatrix} 1+3 & 0+4 \\ 0+5 & 1+6 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 5 & 7 \end{bmatrix}.$$

→ If A & B have same order and $a_{ij} = b_{ij}$ $\forall\, i, j$, then $A = B$.

3. Scalar Multiplication

Multiplying a matrix with a real number.

$A = [a_{ij}]_{m \times n}$. $c \in \mathbb{R}$.

Then,

$$cA = [c a_{ij}]_{m \times n}.$$

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \end{bmatrix}$ $c = 2$.

Then, $cA = \begin{bmatrix} 2\times1 & 2\times2 & 2\times3 \\ 2\times4 & 2\times5 & 2\times6 \\ 2\times3 & 2\times2 & 2\times1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 6 & 4 & 2 \end{bmatrix}$

→ **Linear Combination**

$A_1, A_2, A_3, \ldots, A_p$ are matrices of the same size and $c_1, c_2, c_3, \ldots, c_p$ are real numbers.

$$c_1A_1 + c_2A_2 + c_3A_3 + \cdots + c_pA_p$$

is called a linear combination.

If

$$c_1A_1 + c_2A_2 + c_3A_3 + \cdots + c_pA_p = 0,$$

the linear combination is trivial.

Eg:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 10 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 8\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 10\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. **Matrix multiplication**

$C_{m\times n}$ is the product of $A_{m\times l}$ and

$B_{l \times n}$ if C_{ij} is the dot product of the ith row of A and jth column of B.

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

Each column multiplies with each row to give one entry.

$$B = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} (1\times2)+(2\times3) & (1\times4)+(2\times5) \\ (4\times2)+(5\times3) & (4\times4)+(5\times5) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 14 \\ 23 & 85 \end{bmatrix}.$$

Properties : ① In most cases, $A \cdot B \neq B \cdot A$.

② If $E \cdot B = E \cdot C$, then it does not necessarily mean $B = C$.

③ If $E \cdot D = 0$, then it does not necessarily mean that either $E = 0$ or $D = 0$

④ Any matrix multiplied with identity is that matrix itself.

$$A_{n\times n} \cdot I_n = A_{n\times n} = I_n \cdot A_{n\times n}$$

5. Any matrix multiplied with it's inverse gives you the identity.

$$A_{n\times n} \cdot A^{-1}_{n\times n} = I_n = A^{-1}_{n\times n} \cdot A_{n\times n}.$$

More on this next time.

Elementary Row operations

We can perform row operations to transform $A \to B$.

(a) Interchange two rows.

$$\begin{bmatrix} - R_1 - \\ - R_2 - \\ - R_3 - \end{bmatrix} \underset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} - R_3 - \\ - R_2 - \\ - R_1 - \end{bmatrix}$$

(b) Replace a row with a sum or difference of that row and another row

$$\begin{bmatrix} - R_1 - \\ - R_2 - \\ - R_3 - \end{bmatrix} \underset{R_2 \to R_2 + R_3}{\sim} \begin{bmatrix} - R_1 - \\ - R_2 + R_3 - \\ - R_3 - \end{bmatrix}$$

(c) Scale a row with a nonzero constant.

$$\begin{bmatrix} - R_1 - \\ - R_2 - \\ - R_3 - \end{bmatrix} \underset{R_1 \to R_1 \cdot c}{\sim} \begin{bmatrix} - cR_1 - \\ - R_2 - \\ - R_3 - \end{bmatrix}$$

A is said to be the row equivalent of B if B can be obtained by a sequence of row operations on A.

Row - Echelon Form (REF)

① All non zero rows are above any row of all zeroes.

② Each leading entry of a row is in the column to the right of the leading entry of the row above it.

③ All entries in columns below a leading entry are 0.

To get REF, satisfy ①, ②, ③.

Reduced Row - Echelon Form (RREF)

④ The leading entry in each nonzero row is 1

⑤ Each leading 1 is the only nonzero entry in its column.

Satisfy ①, ②, ③, ④, ⑤ to get RREF.

eg: REF:
$$\begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

RREF : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Pivots.

A pivot position in a matrix is the location of leading 1 in the RREF form of A.
The column containing pivot position is called pivot column.

Tutorial 2

For Elementary row operations, REF and RREF forms, refer to tutorial 1 notes.

System of linear equations

An equation of the form $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ is called a linear equation in n variables. $x_1, x_2, \dots x_n$. $a_1, a_2, \dots a_n$ are coefficients and are numbers.

Eg: $2x + 7y - 10z = 30$ is a linear equation.

A system of linear equations is a set of linear equations involving the same variables.

Eg:
$$\begin{aligned} 2x - 8y + 30z &= 10 \\ x + 20y - \left(\frac{8\sqrt{2}}{\pi}\right)z &= 2e^2 \end{aligned}$$

This is a system of linear equations because
① They involve the same variables, i.e x, y, z
② The powers on x, y, z is 1, so they are linear.

Solution set

The set of numbers that solve each equation in the system (ie upon substitution in place of unknown variables, they make the equation a true statement) is called the solution set of the linear system.

In systems of two variables, each equation represents a straight line and the intersection of these lines is the solution set.

Eg: $$\left.\begin{aligned} 2x - 3y &= 10 \\ 3x + 4y &= 5 \end{aligned}\right\}$$ The system has the solution $x=5, y=0$ because it satisfies each equation, and $(5,0)$ is the intersection of both lines on the x-y plane.

In 3 dimensions, this corresponds to the intersection of planes.

A linear system can have either

consistent:
① 1 unique solution (the intersection of two lines, eg: $(0,0)$ for $\begin{matrix} x-y=0 \\ x+y=0 \end{matrix}$)
② Infinitely many solutions (overlap of two lines. eg: $x+y=2$, $3x+3y=6$)

inconsistent: ③ No solution (The two lines are parallel. eg: $x+y=2$, $x+y=10$)

Matrix notation

Linear systems can be represented in the form of a matrix equation $AX=B$. If your linear system is

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m$$

Then in matrix form, this can be represented as $AX=B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

For a system $AX = B$, we represent an augmented matrix $A|B$ as

$$A|B = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Two linear systems are said to be equivalent if they have the same solution set. Systems have the same solution set if one of the matrices can be obtained by using row operations on the other, i.e. they are row equivalent.

Methods of solving linear systems in matrix notation

I. Gaussian Elimination ($m = n$)

This method consists of two phases: ① Forward & ② Backward.

① Forward phase

In forward phase, you take the augmented matrix and apply row operations to convert to REF form.

Say your augmented matrix is of the form

$$A|B = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

Apply row transforms to take this to

$$REF(A|B) = \left[\begin{array}{cccc|c} c_{11} & c_{12} & \cdots & c_{1n} & d_1 \\ 0 & c_{22} & \cdots & c_{2n} & d_2 \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & & \\ 0 & 0 & \cdots & c_{nn} & d_n \end{array}\right]$$

② Backward phase

Take the REF(A|B), convert it back to system of linear equations. Solve the equations to get the final answer.

Say your REF(A|B) is

$$REF(A|B) = \left[\begin{array}{cccccc|c} c_{11} & c_{12} & \cdots & \cdots & \cdots & c_{1n} & d_1 \\ 0 & c_{22} & \cdots & \cdots & \cdots & c_{2n} & d_2 \\ 0 & 0 & \ddots & & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & c_{nn} & d_n \end{array}\right]$$

then this converts to:

$$c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n = d_1$$

$$c_{22}x_2 + \cdots + c_{2n}x_n = d_2$$

$$\vdots$$

$$c_{nn}x_n = d_n$$

Solve for x_n first, substitute in the above equation to find x_{n-1} and repeat the process till you get x_1.

Cases you might encounter:

<u>No solution</u> $c_{nn} = 0$, $d_n \neq 0$

$\Rightarrow 0 \cdot x_n = d_n$

x_n has no solution. Thus the entire system won't have a solution.

<u>Infinite solutions</u>: $c_{nn} = 0$, $d_{nn} = 0$.

$\Rightarrow$ $0x_n = 0$

This will probably lead you to infinite solutions for one of your variables, thus giving you infinite solutions for the system.

eg: $\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ This will give you $x_3 = 3$, $x_2 = -3$, $x_1 = ?$ (x_1 can be anything)

This gives you infinite solutions.

Unique solution: $c_{nn} \neq 0$

$\Rightarrow x_n = d_{nn}/c_{nn}$.

You will end up with a unique solution.

II. Gauss - Jordan Elimination $(m = n)$

This method only consists of one forward phase, which is to take your Augmented matrix $A|B$ to its RREF form.

Since in your augmented matrix is now in RREF form, it will look something like this:

$$\text{RREF}(A|B) = \left[\begin{array}{ccccc|c} c_{11} & 0 & \cdots & \cdots & 0 & d_1 \\ 0 & c_{22} & \cdots & \cdots & 0 & d_2 \\ 0 & 0 & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & c_{nn} & d_n \end{array}\right]$$

where the c_{ii} is either 1 or 0.

that is $c_{11}, c_{22}, \ldots\ldots c_{nn}$ is either 1 or 0.

The solution is simply then:

$$\left.\begin{aligned} c_{11}x_1 &= d_1 \\ c_{22}x_2 &= d_2 \\ &\vdots \\ c_{nn}x_n &= d_n \end{aligned}\right\} \quad c_{ii} \text{ is either 1 or 0.}$$

Again you have the same cases as above.

No solution: $c_{nn} = 0$, $d_n \neq 0$ $\qquad$ Unique soln: $c_{nn} = 1$

$\therefore x_n$ has no solution. $\qquad$ $x_n = d_n$.

Infinite solutions $c_{nn} = 0$, $d_n = 0$

$\therefore$ one or more of x_i has ∞ solutions.

III. Gaussian elimination ($m < n$)

Used when you have more variables than equations.

You take the $A|B$ to RREF. You will end up with a matrix like

$$RREF(A|B) = \left[\begin{array}{cccccc|c} c_{11} & 0 & & 0 & & c_{1n} & d_1 \\ 0 & c_{22} & \cdots & 0 & \cdots & c_{2n} & d_2 \\ 0 & 0 & \ddots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & 0 & \cdots & \vdots & \vdots \\ 0 & 0 & & c_{mn-1} & & c_{mn} & d_n \end{array}\right]$$

That is, you will have one free variable and $n-1$ pivots. Then you will have ∞ solutions with the solution in terms of your free variable.

Example:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 4 & 5 & 6 & 10 \end{array}\right]$$

$$\overset{RREF}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -5/3 \\ 0 & 1 & 2 & 10/3 \end{array}\right]$$

x_1 and x_2 have pivots. x_3 is free.

Let $x_3 = t$. Then,

$$x_1 - t = -5/3 \Rightarrow x_1 = -5/3 + t$$

$$x_2 + 2t = 10/3 \Rightarrow x_2 = 10/3 - 2t$$

$\therefore$ Solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5/3 + t \\ 10/3 - 2t \\ t \end{bmatrix}, \quad t \in \mathbb{R}.$$

IV. Inverse matrix method

The inverse matrix of a square matrix A is denoted by A^{-1}.

Then $AA^{-1} = I$ and $A^{-1}A = I$. (like how $2 \times \frac{1}{2} = 1$ or $2^{-1} \times 2 = 1$).

Definition: Matrix A is said to be invertible if there is an $n \times n$ matrix C such that $AC = I = CA$. C is called the unique inverse of A and denoted by A^{-1}.

If A is 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then inverse is $A^{-1} = \dfrac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

For all other matrices, for now, to find A^{-1}, use the following method.

We know $AA^{-1} = I$. Augment A and I.

$$A \mid I = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & \cdots & \cdots & 1 \end{array}\right]$$

Row reduce till you end up with the RREF.
If inverse exists, then the left side becomes I and the matrix you get on the right is your inverse.

Inverse of a matrix may or may not exist.
If it doesn't exist, A is called singular or non invertible
If it exists, A is non-singular or invertible.

Note: There is a much easier way to compute A^{-1}. We will see that next tutorial. The above method is called the elementary matrix method.

If inverse exists, then, for a system

$$AX = B,$$

the solution is given by

$$X = A^{-1}B.$$

Note: This always leads to unique solutions.
If inverse doesn't exist, there are most likely no solutions or ∞ many solutions.

Use the Gaussian elimination or Gauss-Jordan elimination to find the solutions in that case.

Tutorial-4 (Tutorial 3 - Quiz)

Determinant

The determinant function of a square matrix A is denoted as det A, it is equal to the sum of all signed elementary products from A. det A is also called the determinant of A.

Elementary product of $A_{n\times n}$ is a product of n entries of matrix A no two of which come from the same row or column.

Eg: 2x2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$a_{11}a_{22}$ and $a_{21}a_{12}$ are the elementary products

Signed elementary products are $+a_{11}a_{22}$ and $-a_{21}a_{12}$.

$$\therefore \det A = +a_{11}a_{22} - a_{21}a_{12}.$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\det A = +(1.4) - (2.3)$$
$$= 4 - 6 = -2.$$

Eg: 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Signed elementary products are

$+a_{11}a_{22}a_{33}$, $+a_{12}a_{31}a_{23}$, $+a_{13}a_{21}a_{32}$.

$-a_{11}a_{32}a_{23}$, $-a_{12}a_{21}a_{33}$, $-a_{13}a_{31}a_{22}$

$$\therefore \det A = +a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{12}a_{21}a_{33} - a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$$

$$= a_{11}(+a_{22}a_{33} - a_{32}a_{23}) - a_{12}(+a_{21}a_{33} - a_{31}a_{23}) + a_{13}(+a_{21}a_{32} - a_{31}a_{22}).$$

Notice the similarity of the brackets and the sum of the signed elementary products of a 2×2 matrix! Especially $a_{22}a_{33} - a_{32}a_{23}$. Is it not similar to $a_{11}a_{22} - a_{21}a_{12}$? And what is the sum of the signed elementary products of a 2×2 matrix. It is it's determinant. So, we can say that the determinant of a 3×3 matrix is nothing but the sum of a_{ij} of any single row/column multiplied by the determinant of the submatrix remaining after the deletion of the ith row and jth column, and multiplying $(-1)^{i+j}$ to that product. What we have come upon is the cofactor expansion method of finding determinants of an n×n matrix.

The cofactor expansion method

Minor: A minor of an entry a_{ij} in a matrix A is denoted by M_{ij} and is defined to be the determinant of submatrix that remains after the deletion of the ith row and the jth column from matrix A.

Cofactor: The number $(-1)^{i+j} M_{ij}$ is called the cofactor of entry a_{ij}.

eg: $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

$M_{21} = 4(9) - 6(7) = 36 - 42 = -6$

$C_{21} = (-1)^{2+1} M_{21} = (-1)(-6) = \underline{\underline{6}}.$

Then, the determinant of a square matrix can be found as a sum of products between their entries and cofactors, in any row or column of A.

Eg:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{bmatrix}$$

$$\det A = 1(1) - 2(1) + 3(-3) - 4(-4)$$
$$= 1 - 2 - 6 + 16 = \underline{\underline{9}}.$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 0 & 1 \end{vmatrix} = 2 \qquad \begin{vmatrix} 2 & 1 & 0 \\ 3 & 0 & 0 \\ 4 & 0 & 1 \end{vmatrix} = -3 \qquad \begin{vmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ 4 & 0 & 0 \end{vmatrix} = -4$$

Inverse matrices – Cofactor method

Adjoint matrix: The transpose of the matrix of cofactors of a matrix A is called the adjoint of A, denoted by adj(A).

Cofactor matrix: Matrix of cofactors, a $n \times n$ matrix. Each entry C_{ij} correspond to the cofactor of a_{ij} of matrix A.

Eg: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

$$\therefore \operatorname{adj}(A) = C^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Then, the inverse is given by

$$\boxed{A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)}$$ //.

That is, take the adjoint matrix of A, divide it by the determinant of A, and you have your A^{-1}.

Ex. Inverse of a generic 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$ Then, $\det A = ad - bc$.

$$C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}. \therefore \operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\therefore \boxed{A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

You can use this method to solve linear systems with matrices that are invertible. If $AX = B \Rightarrow X = A^{-1}B$ (Note: Donot do BA^{-1}. That is incorrect). This only works for $m=n$.

Cramer's method of solving linear systems

It is a very efficient method to find solutions to systems of equations with equal equations and variables ($m=n$).

It states that the set of solutions for an arbitrary system of n equations and n variables

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{n1}x_1 + \cdots\cdots + a_{nn}x_{nn} = b_n \end{array}\right\} AX = B$$

is given by $x_1 = \frac{\Delta_1}{\Delta}$, $x_2 = \frac{\Delta_2}{\Delta}$, ... $x_n = \frac{\Delta_n}{\Delta}$.

$\Delta = \det A$. and

$$\Delta_i = \begin{vmatrix} a_{11} & a_{12} \cdots & \cdots b_1 & \cdots a_{n1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & b_n & a_{nn} \end{vmatrix}.$$ That is, Δi is the determinant of a matrix A where the ith column of A is replaced with B.

Eg: $x + y = 3$ $\quad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$x - y = 4$

$\Delta = -1 - 1 = -2.$ $\quad \Delta_1 = \begin{vmatrix} 3 & 1 \\ 4 & -1 \end{vmatrix} = -3 - 4 = -7$

$\Delta_2 = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 4 - 3 = 1.$

$\therefore \quad x = \frac{\Delta_1}{\Delta} = \frac{-7}{-2} = 7/2. \quad y = \frac{\Delta_2}{\Delta} = \frac{1}{-2} = \underline{\underline{-\frac{1}{2}}}.$

Eigenvalues and Eigenvectors

When we multiply a vector to a matrix, the result is usually the original vector stretched and rotated.

Sometimes, when you take a vector and multiply it to a matrix, the matrix only stretches it. The factor by which it stretches is called eigenvalue and the vector is called an eigenvector.

Eigenvalues and eigenvectors are important and unique characteristics of a matrix, and have several applications. They can tell you the stability of a system, can help you describe a quantum state, etc etc. The word "eigen" comes from German and means self or own.

Definition: A nonzero vector $\vec{v}$ in $\mathbb{R}^n$ is called the eigenvector of A if $A\vec{v} = \lambda\vec{v}$ for some scalar λ. The scalar λ is called the eigenvalue of A corresponding to $\vec{v}$.

Characteristic equation

If $A\vec{x} = \lambda\vec{x} \Rightarrow (A - \lambda I)\vec{x} = \vec{0}$. I here is the identity matrix.

This will have non trivial solutions iff $(A - \lambda I)$ is not invertible.

So, $\det(A - \lambda I) = 0$. This is the characteristic equation.

$$\det(A-\lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow C_n \lambda^n + C_{n-1}\lambda^{n-1} + \cdots + C_1\lambda + C_0 = 0.$$

Solve equation to find λ.

Note: $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$ (ie the product of all eigenvalues).

Eigenspace

Once you have all the eigenvalues, now you want to find the eigenvectors for each eigenvalue.

For that, we know $(A-\lambda I)\vec{x} = 0$. Substitute for λ, and then solve for $\vec{x}$ using Gaussian elimination or Gauss-Jordan elimination. Note, you will almost always get infinitely many solutions.

Def: The set of all eigenvectors (aka all solutions to $(A-\lambda I)\vec{x} = 0$) is called the eigenspace of matrix A corresponding to λ.

Linear independance

A set of vectors are called linearly independent if the solution to

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$$ is the trivial solution.

That is only $a_1 = a_2 = \cdots = a_n = 0$ is the solution.

Iterated mapping

Let A be a 2×2 matrix with two linearly independent eigenvectors and 2 real eigenvalues. Then,

A maps $v_1 \longrightarrow \lambda v_1$ $\qquad \lambda_1^2 v_1 \longrightarrow \lambda^3 v_1$

$\lambda_1 v_1 \longrightarrow \lambda_1^2 v_1$ $\qquad$ and so on.

If $\vec{v}$ is not an eigenvector, then to get the third mapping we would have to multiply A 3 times by itself.

However, if we say $\vec{v} = a\vec{v}_1 + b\vec{v}_2$ (ie a linear combination of v_1 and v_2), then

$$A^n \vec{v} = a \lambda_1^n v_1 + b \lambda_2^n v_2 .$$

Example: Leslie matrix <u>example</u> in lecture notes.